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Robustness of Solutions to a Benchmark Control Problem

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The robustness of 10 solutions to a benchmark control design problem presented at the 1990 American Control Conference has been evaluated. The 10 controllers have second- to eighth-order transfer functions and have been designed using several different methods, including H_∞ optimization, loop-transfer recovery, imaginary-axis shifting, constrained optimization, structured covariance, game theory, and the internal model principle. Stochastic robustness analysis quantifies the controllers' stability and performance robustness with structured uncertainties in up to six system parameters. The analysis provides insights into system response that are not readily derived from other robustness criteria and provides a common ground for judging controllers produced by alternative methods. One important conclusion is that gain and phase margins are not reliable indicators of the probability of instability. Furthermore, parameter variations actually may improve the likelihood of achieving selected performance metrics, as demonstrated by results for the probability of settling-time exceedance.

Introduction

CONTROL systems should be designed to maintain satisfactory stability and performance characteristics not only at nominal operating points but over a range of parameters that encompasses system uncertainty. These systems should be robust, but there is a limit. Unbounded robustness is no more attractive than inadequate robustness, because nominal performance and insensitivity to parameter variations tend to produce conflicting design requirements. Hence, the degree of robustness that must be furnished for satisfactory operation is related to the system variations that are most likely to occur.

Measures of robustness should be easily understood and should be directly connected to control design objectives. They should be consistent with what is known about the structure and parameters of the plant's dynamic model. These goals are best served when robustness is expressed in terms of the likelihood that commonly accepted properties fall within acceptable bounds and when parameter variations are expressed in terms of readily measured system specifications. A method of satisfying these evaluation criteria is presented here.

This paper demonstrates the application of stochastic robustness analysis (i.e., determining the probability of unsatisfactory stability or performance resulting from expected parameter uncertainty) to solutions of the 1990 American Control Conference Benchmark Control Problem.[1] Stochastic robustness is seen to provide a useful, unifying analytical framework that is intuitive and has a direct, physical meaning.

Description of the Problem

The benchmark plant is a dual-mass/single-spring system with noncollocated sensor and actuator[1]; its state-space model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} w \tag{1}$$

$$y = x_2 + v \tag{2}$$

$$z = x_2 \tag{3}$$

where x_1 and x_2 are the positions of the masses, x_3 and x_4 are their velocities, and u is a control force on m_1. The plant is disturbed by w on m_2, and the measurement of x_2 is corrupted by noise v in y. The corresponding actuator and disturbance input/output transfer functions are

$$\mathcal{H}_{uy} = \frac{(k/m_1 m_2)}{s^2\left[s^2 + k(m_1+m_2)/m_1 m_2\right]} \tag{4}$$

$$\mathcal{H}_{wy} = \frac{(1/m_2)(s^2 + k/m_1)}{s^2\left[s^2 + k(m_1+m_2)/m_1 m_2\right]} \tag{5}$$

The plant has eigenvalues at $(\pm j\sqrt{k(m_1+m_2)/m_1 m_2}, 0, 0)$ and is undamped. A single-input/single-output (SISO) controller must close its loop around $\mathcal{H}_{uy}$, which has a pole-zero surplus of 4. The high-gain asymptote of at least one root lies in the right half plane for any SISO feedback compensator that has fewer than two surplus zeros. Because the open-loop roots are on the imaginary axis, the magnitudes of root departure angles must exceed 90 deg if marginal instability is to be avoided at low loop gain.

Three design problems are posed in Ref. 1. Benchmark problem 1 (BP-1) requires 1) a 15-s settling time for unit disturbance impulse and nominal mass-spring values ($m_1 = m_2 = k = 1$) and 2) closed-loop stability for fixed values of mass and $0.5 < k < 2$. The second problem, BP-2, replaces the unit-impulse disturbance by a sinusoidal disturbance with 0.5-rad/s frequency but unknown amplitude and phase. Asymptotic rejection of the signal should be achieved with a 20-s settling time for nominal masses and $0.5 < k < 2$. The third problem, BP-3, is like BP-1, except that m_1, m_2, and k are uncertain with mean values of 1 and unspecified bounds. A number of additional problem specifications are left to the discretion of the designer. For example, it is presumed that a noise model

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$v(t)$ would be considered, but details of the model are open. Subjective goals include achieving reasonable performance/stability robustness, minimizing controller effort, and minimizing controller complexity.

Design Solutions and Nominal Performance

Five papers containing design solutions appear in the American Control Conference Proceedings,[2-6] one paper became available after the conference,[7] and additional designs were obtained from the authors. The transfer functions for these controllers are presented in the Appendix. Fixed-order compensators achieving approximate loop-transfer recovery are motivated in Ref. 2, leading to designs A–C. An H_∞ plus $j\omega$-axis shifting approach is taken in Ref. 3, producing design D. Reference 4 uses nonlinear constrained optimization to produce design E. Structured covariance terms are added to the linear quadratic Gaussian (LQG) algebraic Riccati equations to generate design F in Ref. 5. Design G is a game-theoretic controller based on linear exponential Gaussian and H_∞ concepts and is discussed in Ref. 6. H_∞ controllers using the internal model principle are presented in Ref. 7 (designs H–J). G and J are designed to reject the sinusoidal disturbance (BP-2) rather than the unit impulse disturbance (BP-1). All but two of these designs (A and D) contain non-minimum-phase zeros. The benchmark criteria do not address command-input responses; hence, the initially reversed time response of systems with an odd number of non-minimum-phase zeros is not penalized. Design G has an even number of right-half-plane zeros, which would not produce reversed response.

The problem statement contains an ambiguity that could have affected the designers' interpretations of satisfactory response. Settling time is normally defined as an attribute of unit-step-function response. For example, Ogata[8] states that "The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%)." For a second-order system the 2% settling time can be precisely calculated as $4/\zeta\omega_n$, where ζ is the damping ratio and ω_n is the natural frequency of the oscillatory mode. However, Takahashi et al.[9] found that "Exact analytical expressions for ... settling time become prohibitively complicated for systems of order higher than two." The benchmark ambiguity is that the final value of a strictly stable impulse response (BP-1) is zero; hence, there is no steady-state value on which to base percentage response.

Nominal performance characteristics of the controllers are summarized in Table 1, which presents compensator numerator and denominator order (Num Ord and Den Ord), two definitions of settling time (T_S^* and T_S^{**}), maximum control usage (u_{max}) resulting from a unit w disturbance, gain margin (GM), phase margin (PM), output response to 0.5/rad/s sinusoidal disturbance (SR), and covariance of control response (U_{cov}) to measurement noise (v) with unit standard deviation. All compensators are proper (the number of zeros does not exceed the number of poles), but three (C, D, and E) are not strictly proper (the number of zeros equals the number of poles). Hence, designs A, B, and F–J can be classified as low-pass filters, whereas designs C–E do not roll off at high frequencies.

T_S^* portrays the settling time as the time for which x_2 is captured within a 0.1-unit envelope about its zero steady-state value, given an initial unit w disturbance impulse. T_S^{**} is based on the damping ratio and natural frequency of the dominant mode and is calculated as $4/\zeta\omega_n$. Neither of these definitions adheres to the conventional definition, but each has its merits. T_S^* is consistent with the BP-1 problem specification, in that it reflects a response to a unit w disturbance; however, it is amplitude dependent. T_S^{**} is independent of amplitude, but it is unrelated to the disturbance input and is not an accurate portrayal of the full system's settling time in response to a unit step input. Table 1 indicates that only three of the compensators satisfy a 15-s criterion by the first definition, whereas six compensators have settling times of ≤ 15.2 s by the second definition.

Four compensators use measurably more control than the others in responding to the disturbance. Increasing gain margin generally is accompanied by increasing phase margin for these 10 designs (Fig. 1), although the relationship is not monotonic. With the exception of design D, stability margins are less than the 8-dB/30-deg rules of thumb (e.g., Ref. 10) often suggested as design goals for SISO systems. Sinusoidal disturbance rejection of most controllers is similar, although design D's response is an order of magnitude smaller. Designs G and J, specifically intended to reject a 0.5-rad/s sinusoid, eliminate the disturbance completely in the steady state. (The settling time in achieving this response was not evaluated.) The noise-response covariance of the control is generally proportional to its peak disturbance-impulse response for strictly proper compensators. The three non-strictly proper compensators have infinite control covariance in response to continuous white measurement noise v (with infinite bandwidth).

Stochastic Robustness Analysis

Stochastic robustness analysis (SRA) is based on a statistical portrayal of parameter variations and their effects. If parameters take a finite number of discrete values, each with known

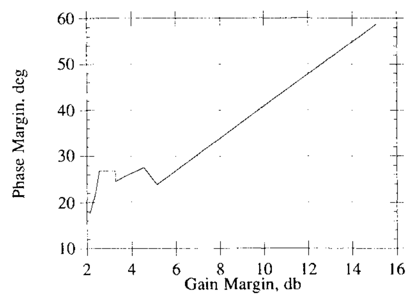

Fig. 1 Nominal gain and phase relationships of the 10 controllers.

Table 1 Nominal characteristics of 10 controllers

Design	Num Ord	Den Ord	T_S, s*[a]	T_S, s**[b]	u_{max}	GM, db	PM, deg	SR, db	U_{cov}
A	2	3	21.0	14.8	0.514	2.56	26.7	10.1	6.30
B	2	3	19.5	15.2	0.469	3.27	26.8	13.2	13.02
C	2	2	19.7	15.2	0.468	3.27	26.5	13.3	∞
D	4	4	9.9	8.8	297.8	15.10	58.7	1.47	∞
E	2	2	18.2	8.01	0.884	2.39	22.0	17.1	∞
F	3	4	13.7	22.0	2.397	5.15	23.8	13.4	6×10^4
G	5	8	31.3	35.7	1.458	3.61	25.4	$-\infty$	173.5
H	3	4	14.9	11.9	0.574	3.28	24.5	14.9	2.48
I	3	4	17.8	17.2	0.416	4.56	27.5	13.3	0.95
J	5	6	43.2	23.8	1.047	2.14	17.5	$-\infty$	77.42

[a] *Defined for 0.1-unit x_2 response envelope for unit-impulse w.
[b] **Defined by $4/\zeta\omega_n$ (provided by B. Wie).

or estimated probability, the analysis can be based on a finite number of function evaluations, and the probabilistic result is exact (within the accuracy and precision of problem modeling). If the parameters are continuous or the number of finite combinations is too large for practical computation, Monte Carlo evaluation can be used to estimate probabilities within arbitrarily small confidence intervals. If a binary judgment can be made of function values (e.g., satisfactory/unsatisfactory or stable/unstable), then the corresponding probability distribution is binomial, and confidence intervals are readily estimated from the number of function evaluations (e.g., Ref. 11). Further details of SRA can be found in Refs. 12–17.

Test Cases for the Benchmark Problem

Uncertain parameters are assumed to have continuous, bounded, uniform, and uncorrelated probability distributions for this analysis. (The original problem identifies uncertain parameters and their bounds, making no statement about distributions.[1]) Three increasingly demanding sets of parameter uncertainties are used to test the controllers. The first two are specified in Ref. 1, and the third is new.

Problem E-1: $0.5<k<2$, all other parameters take nominal values, as in BP-1 and BP-2.

Problem E-2: $0.5<k<2$, $0.5<m_1<1.5$, and $0.5<m_2<1.5$, as in BP-3. Reference 1 does not specify limits on m_1 and m_2; values of $\pm 50\%$ are adopted here.

Problem E-3: Same as E-2; in addition, $0<c<0.1$, $0.9<f<1.1$, and $0.001<\tau<0.4$ s, where c represents internal damping between the masses; f is loop-gain uncertainty due to multiplicative variation in observation, control gain, or actuator effectiveness; and τ is the time constant for a first-order lag between controller command and actuator response. Uncertainty in the damping ratio c increases open-loop damping, and the time lag is always greater than the nominal value of zero.

With all six parameters, the state-space model for E-3 becomes

$$\dot{x}' = F'x' + G'u_c + L'w \tag{6}$$

where x' is defined as $[x_1\ x_2\ x_3\ x_4\ u]^T$, and

$$F' = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -k/m_1 & k/m_1 & -c/m_1 & c/m_1 & f/m_1 \\ k/m_2 & -k/m_2 & c/m_2 & -c/m_2 & 0 \\ 0 & 0 & 0 & 0 & -1/\tau \end{bmatrix} \tag{7}$$

$$G' = [0\ \ 0\ \ 0\ \ 0\ \ 1/\tau]^T \tag{8}$$

$$L' = [0\ \ 0\ \ 0\ \ 1/m_2\ \ 0]^T \tag{9}$$

The compensators are modeled by

$$\dot{x}_c = Ax_c + By \tag{10}$$

$$u_c = Cx_c + Dy \tag{11}$$

where x_c is the compensator state; u_c is the actuator command; A, B, C, and D are the compensator matrices; and y is x_2.

Performance Metrics for the Benchmark Problem

Robustness is best characterized by problem-dependent metrics that have a direct bearing on the measurable stability and performance of the system. Here, they portray the likelihood that classical stability bounds will be exceeded, that settling time will not be achieved, and that control usage will exceed acceptable values. For demonstration of SRA, parameter uncertainties are represented by uniform distributions within arbitrary (but reasonable) bounds. In practical application, the control-system designer would have similar, problem-specific specifications to meet.

Each of the following probabilistic performance metrics has a binomial distribution and is estimated using Monte Carlo evaluation. Uniform, bounded parameters are calculated by random-number generators according to the specifications of the previous section. The associated binomial confidence level depends on the number of evaluations and the value of the probability estimate.[13] Each estimate is the result of 20,000 evaluations; for a probability estimate of 0.1, the 95% confidence interval would be ± 0.004. The performance metrics are:

1) P_I: Probability of instability. This probability portrays the likelihood that parameter variations force at least one closed-loop root into the right half plane.

2) P_{T_s}: Probability of settling-time exceedance. This probability is derived from a time-history calculation with a unit-impulse w input (i.e., based on T_S^*) and estimates the likelihood that the actual response of z will fall outside a ± 0.1-unit envelope after 15 s.

3) P_u: Probability of control limit exceedance. This probability corresponds to the requirement in Ref. 1 to minimize controller effort. It is the probability that peak actuator displacement will exceed a saturation limit in response to a unit disturbance (w) impulse. The saturation limit was chosen to be one unit for this analysis.

4) P_f: Probability of unsatisfactory sinusoidal disturbance rejection. This probability involves the likelihood that the amplitude of steady-state z response exceeds one unit with a unit sinusoidal disturbance at 0.5 rad/s.

Computation times indicate that current workstations are fast enough to execute practical SRA, and massively parallel computers could provide interactive turnaround. For the typical closed-loop system considered here, roughly 900 sets of eigenvalues were generated per minute per million floating-point operations per sec (MFLOP). This is drawn from compiled Pascal code executed on a 0.9-MFLOP Silicon Graphics 4D/20 workstation. The complete evaluation was computed at a rate of 30 sets/min/MFLOP using MATLAB on a Macintosh IIx computer. At these rates, a 5000-MFLOP parallel computer (e.g., 64K CM-2 Connection Machine) would evaluate 20,000 sets of eigenvalues in 0.25 s, and the full evaluation would take about three times longer.

Results of the Analysis

The results of the SRA indicate a wide range of characteristics in the 10 controllers. This reflects varying emphasis in satisfying the problem specifications, as well as significant differences in compensator order and design philosophy. It should be emphasized that none of the controllers was designed for the express purpose of satisfying SRA criteria, and it is likely that each design approach could be fine-tuned to produce better results than those shown here. Using criteria that have high engineering significance, SRA provides a "level playing field" on which to judge the robustness of controllers that were designed by alternative methods. Tables 2–4 present results, with maximum probabilities for each evaluation problem indicated by bold letters and minimum values underlined.

Probability of Instability

For the least uncertain case (E-1), over half of the controllers are estimated to have zero probability of instability, whereas design A has a 16% likelihood of instability (Table 2). With increasing parameter uncertainty (E-2 and E-3), all controllers have nonzero P_I. The probability of design A is essentially unchanged, and design J becomes the controller most likely to be unstable.

It is interesting to compare the probabilities of instability on the bases of gain and phase margins, quantities often assumed to indicate the robustness of SISO systems. Figures 2 and 3 demonstrate that nominal values of GM and PM are *not* good predictors of P_I. (Note that these bar charts present results for the 10 compensators; hence, GM and PM are not evenly dis-

Table 2 Probability of instability

Design	E-1	E-2	E-3
A	**0.160**	0.159	0.165
B	0.023	0.042	0.039
C	0.021	0.040	0.041
D	0.000	0.004	0.059
E	0.000	0.097	0.125
F	0.000	0.119	0.224
G	0.000	0.203	0.232
H	0.000	0.046	0.099
I	0.000	0.013	0.029
J	0.039	**0.237**	**0.245**

Table 3 Probability of settling-time violation

Design	E-1	E-2	E-3
A	0.971	0.962	0.793
B	**1.000**	0.969	0.963
C	**1.000**	0.968	0.874
D	0.000	0.004	0.072
E	**1.000**	**1.000**	0.999
F	0.633	0.859	0.967
G	**1.000**	0.999	**1.000**
H	0.742	0.909	0.986
I	0.756	0.918	0.986
J	**1.000**	**1.000**	0.968

Table 4 Probability of control-limit exceedance

Design	E-1	E-2	E-3
A	0.160	0.159	0.165
B	0.023	0.043	0.047
C	0.021	0.041	0.041
D	**1.000**	**1.000**	**1.000**
E	0.000	0.391	0.409
F	**1.000**	**1.000**	**1.000**
G	**1.000**	0.886	0.889
H	0.000	0.133	0.162
I	0.000	0.023	0.030
J	0.857	0.542	0.527

tributed.) In most cases, increasing parameter uncertainty increases P_I, but there are no consistent trends with GM and PM. Parameter variations have complex effects on the *shape* of each controller's Nyquist plot, and these effects cannot be portrayed simply by changing loop gain or phase angle.

This result brings into question the utility of transfer-function/return-difference-matrix singular values as measures of the stability robustness of multi-input/multi-output (MIMO) systems. MIMO singular-value analysis is loosely equivalent to SISO gain-margin analysis (e.g., Ref. 18). Arbitrary, real parameter variations have complicated effects on the frequency distributions of MIMO singular values, changing their shapes as well as their magnitudes. Unless the frequency distributions of *nominal* MIMO norms retain their shapes under parameter variation (or follow some predictable pattern), the relationships of nominal maximum or minimum values to allowable bounds tells little about stability robustness. Norm bounds can be reliably evaluated only by considering the norms of *perturbed* systems.

A higher compensation order does not necessarily improve robustness (Tables 1 and 2). The compensators with the most stability robustness are fourth order, and the next most robust controllers are second and third order. Increased nominal control usage, either as a consequence of a disturbance impulse or measurement noise, generally corresponds to decreased stability robustness, although design D provides a significant exception.

Probability of Settling-Time Violation

All but three of the controllers (D, F, and H) exceed the 15-s settling-time objective (defined by T_S^*) in the nominal case (Table 1); hence, it is not surprising that the probability of settling-time violation with parameter uncertainty is high as well (Table 3). Design D provides a notable exception: Its nominal T_S^* is 9.9 s, and P_{T_s} is small for all three evaluation cases. For problem E-1, half of the controllers violate the goal all the time, but two of the controllers with nominal T_S^* above 15 s (H and I) have a considerable likelihood (25%) of satisfying the objective when the spring-constant uncertainty is considered. Further uncertainty (problems E-2 and E-3) reduces the probability of settling-time violation for more controllers, illustrating the counterintuitive result that the effects of uncertainty are not always unfavorable.

Probability of Control Limit Exceedance

The probability of excessive control response to disturbance impulse P_u is shown in Table 4. Over half of the nominal response are within the $u_{\max}$ criterion chosen for this analysis (Table 1). Furthermore, there is an identifiable trend in the relationship between $u_{\max}$ and P_u (Fig. 4). Several controllers (E, H, and I) have zero probability of violating this criterion for problem E-1, and designs B, C, and I retain low values of P_u for all three problems. Designs D and F have 100% P_u in all three cases, which is traceable to very high nominal control usage. Once again, nominally marginal cases (G and J, the two controllers designed for rejection of the sinusoid) exhibit reduced probability of exceedance for problems E-2 and E-3. From Eq. (1), increased m_1 and m_2 decrease the effects of u and w, and added damping (c) and first-order lag (τ) reduce control peaks in some cases, reducing the probability of high control levels.

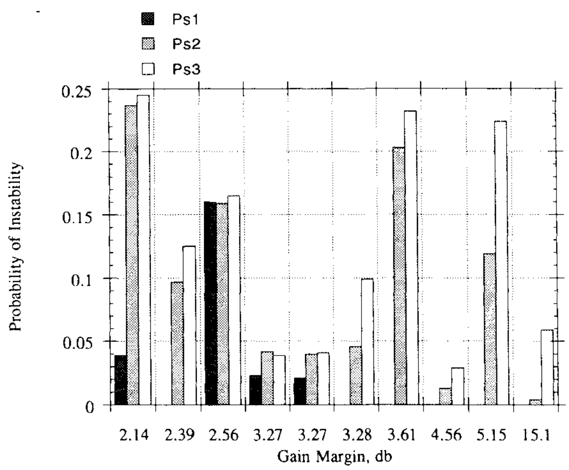

Fig. 2 Probability of instability vs gain margin for three evaluation problems.

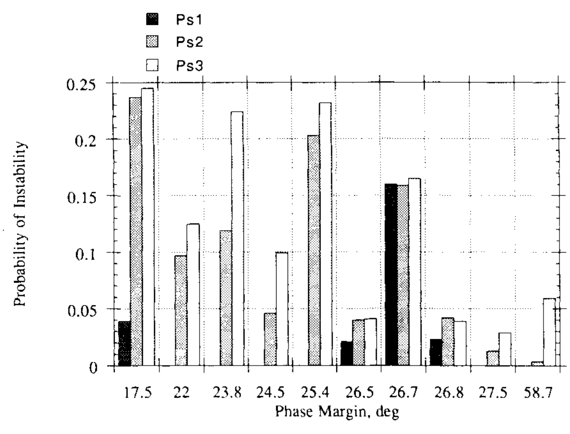

Fig. 3 Probability of instability vs phase margin for three evaluation problems.

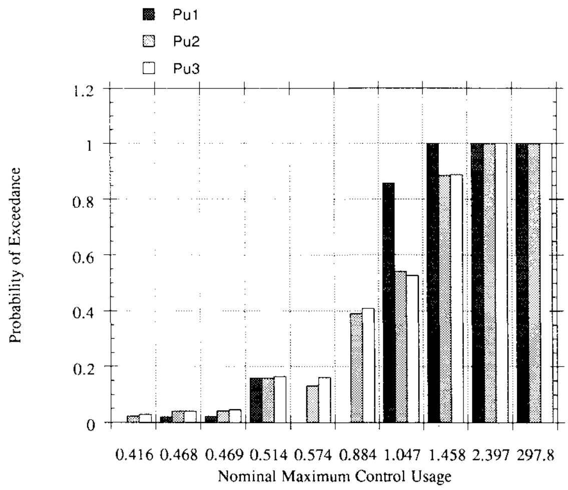

Fig. 4 Probability of control-limit exceedance vs nominal maximum control response to a disturbance impulse.

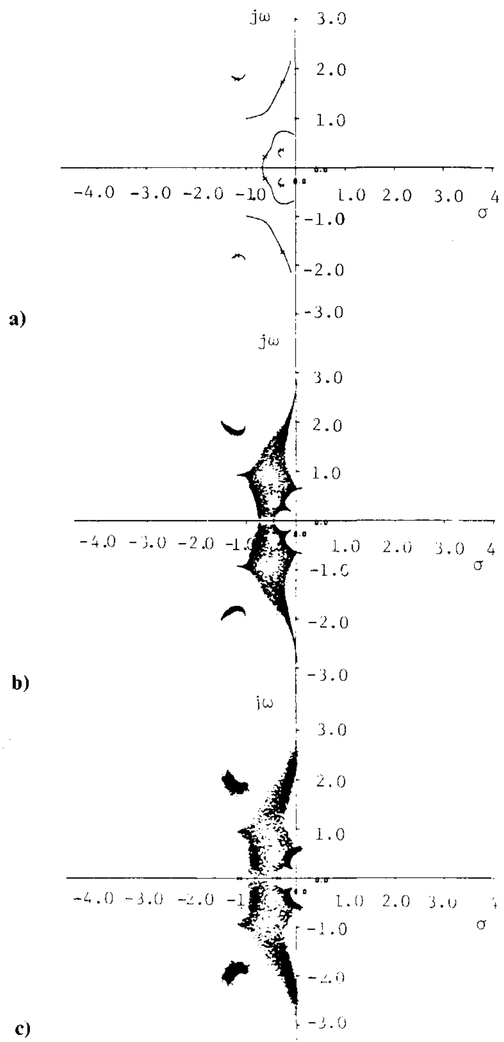

Fig. 5 Stochastic root locus of design H (scatter plot): a) problem E-1; b) problem E-2; c) problem E-3.

Sinusoidal Response Characteristics

When 0 dB is chosen as an upper response limit, the two controllers designed to reject the sinusoid (G and J) do so perfectly ($P_f = 0$), whereas all the others exceed the limit all the time. The transfer functions (Appendix) show that designs G and J effectively "notch" the 0.5-rad/s disturbance-input frequency to produce these results. Without notch filters the remaining controllers cannot give special attention to discrete-frequency inputs, and their frequency response of ~0.5 rad/s always exceeds 0 dB. If the frequency of the sinusoidal disturbance were uncertain, the notch filters could be less effective, but there would be little change in the response of the other controllers.

Stochastic Root Loci and Parametric Histograms

Graphical results give insight into the nature and causes of possible instability. The stochastic root locus is an s-plane plot of the eigenvalues that result from each Monte Carlo evaluation, expressed either as a two-dimensional scatter plot of closed-loop roots or an oblique three-dimensional view of the density of roots within subspaces of the s plane.[13] The former plot is easily generated from the calculations, and the latter has the advantage of showing the distribution along the real axis.[17] In addition, histograms of the parameters associated with instability can be related to origins of the problem.

Scatter plots for design H show the progression of eigenvalue uncertainty from problem E-1 to E-3 (Fig. 5). For prob-

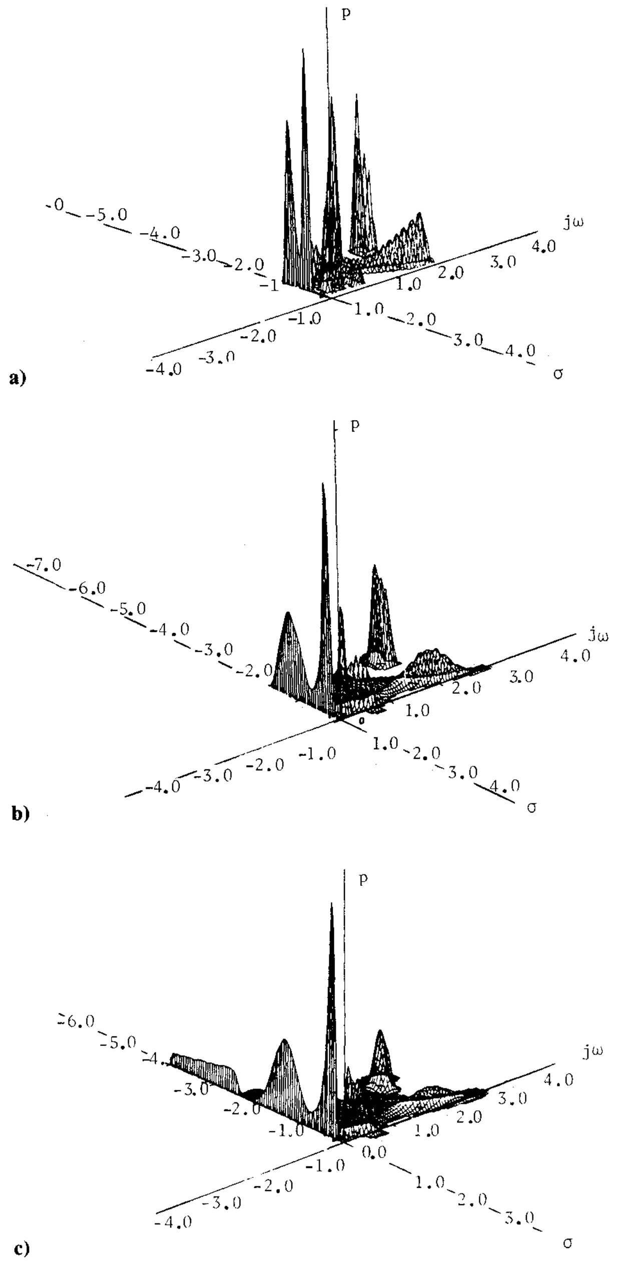

Fig. 6 Stochastic root locus of design H (three-dimensional view): a) problem E-1; b) problem E-2; c) problem E-3.

lem E-1 only a single parameter varies (the spring constant k). The distribution follows the conventional root locus (with nominal closed-loop locations indicated by ×), although the density of roots varies along the curves. The pairs of roots near the origin are most closely associated with the plant, whereas the higher-frequency roots are compensator modes. None of the root loci extend into the right half plane, and P_I is zero (Table 2). Three parameters vary in problem E-2, and the stochastic root locus becomes an areal distribution of roots, some of which extend into the right half plane (Fig. 5b). Because the parameter variations are bounded, there are crisp edges to the distributions. The unstable cusps at 0.6 and 2.6 rad/s can be associated with plant and controller modes. Further parametric uncertainty (problem E-3) broadens the distributions and increases the probability of instability.

The same information is presented in unsmoothed three-dimensional form in Fig. 6 (upper half plane only), which shows the distribution of real roots as well. The three-dimensional representation is especially effective when displayed on a graphics workstation that allows the viewpoint to "fly around" the distribution.

To see which parameter values are associated with instability, the values are recorded whenever the system is found to be unstable. These values are collected in intervals, the number of

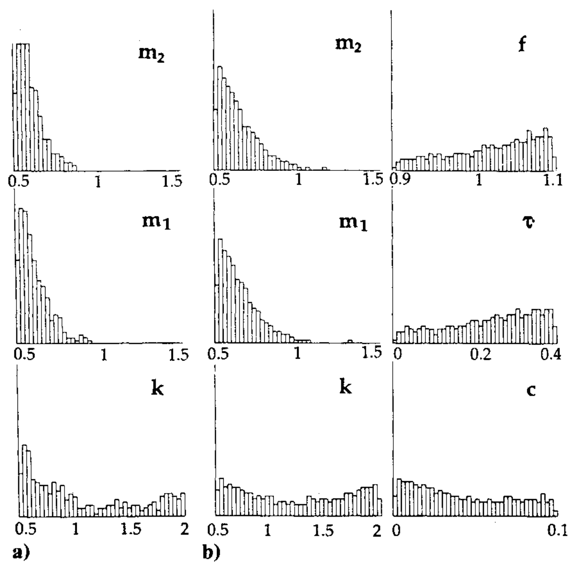

Fig. 7 Parameter histograms for all unstable cases, design H: a) problem E-1; b) problem E-3.

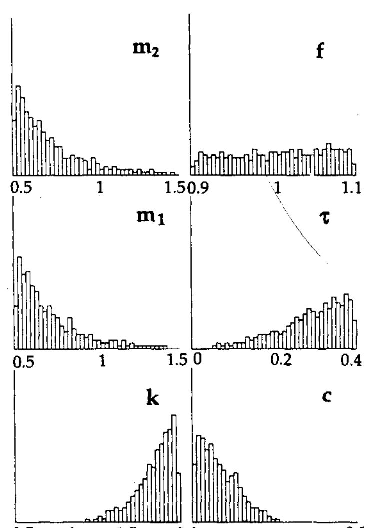

Fig. 8 Parameter histograms for high-frequency unstable cases, design H, problem E-3.

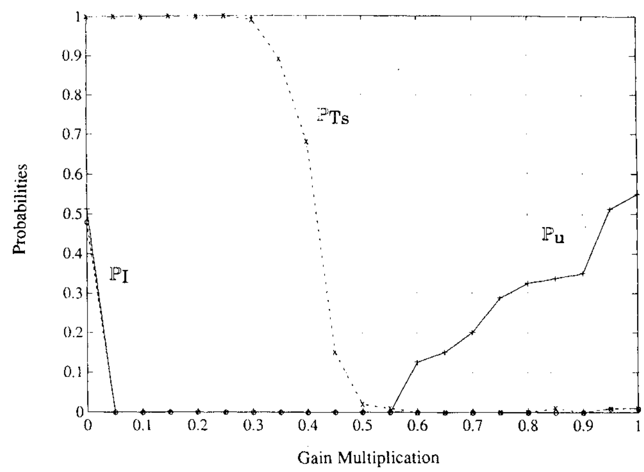

Fig. 9 Effect of reducing design D loop gain on P_I, P_{T_s}, and P_u.

values in each interval is counted, and the resulting histogram provides an estimate of the conditional probability density function for each parameter. If a parameter has little effect on stability, then the histogram should show the same distribution as produced by the random number generator—in this case, a uniform distribution. If particular values of the parameter increase the probability of instability, the histogram has higher values in that region.

For design H and problem E-2, instability often occurs when the masses have low values but never occurs with high values (Fig. 7a). Low mass values increased the probability of instability for all the designs. Extreme values of the spring constant also are associated with instability, low values having the edge in this example.

For problem E-3 (Fig. 7b), the distributions become less crisp, as otherwise unstable values of mass can be stabilized by damping and otherwise stable values of mass can be destabilized by increased loop gain or first-order lag. The spring constant shows a slight bimodal distribution due to the two modes of instability with roots of approximately 0.6 or 2.6 rad/s. This can be seen by recording the parameter values only when the system is found to be unstable *and* the unstable roots have a high frequency. The resulting histograms (Fig. 8) show that there are unstable high-frequency roots only if the spring constant is high and the damping is low. With increased damping, there is no high-frequency instability.

These results can be used in three ways. The probability of instability could be reduced if it were possible to ensure that the plant parameters did not move into the areas that are found to cause problems. This might be the result of improved quality assurance on the important parameters or by shifting the mean of the parameter variation. If it is not possible to affect the actual parameter variations, then the control system could be redesigned using the problematic values of parameters as nominal values. For example, the control system could be redesigned using nominal values of 0.7 for the masses.

A third use of the distributions can occur if one of the varying parameters represents a control design parameter. For instance, if the loop gain f were treated as a design variable, then it is clear that attenuating the gain would reduce the probability of instability. This alternative is demonstrated using design D. It has been seen that design D had generally good robustness but very high actuator use. Peak actuator usage can be reduced by reducing the loop gain, and the effect of gain attenuation on robustness subject to problem E-2 is shown in Fig. 9. For this analysis, only 100 Monte Carlo evaluations were carried out per design point, but the results show clear trends. As the gain is reduced, the probability of control saturation is reduced without significant increase in P_I or P_{T_s} until the attenuation reaches 0.6, when P_{T_s} begins to increase. Reducing the gain further produces a clear tradeoff between the probabilities of control saturation and settling-time violation. References 19 and 20 present similar methods of control system design based on search and statistical evaluation.

Conclusions

Stochastic robustness analysis of 10 controllers designed for the ACC Benchmark Control Problem provides useful quantification of stability and performance sensitivities to parameter variations. The SRA method is flexible and can be tailored to the design requirements and system specifications of particular control problems. Qualitative selection of the best controller depends on the relative importance of several metrics, which are readily described in a probabilistic framework.

Several conclusions can be drawn from this analysis. The analysis shows that gain and phase margins are not good predictors of the relative stability robustness of different SISO controllers, because robustness is tied closely to the actual plant uncertainties and their effects on (implied) Nyquist contours. This result implies that robustness analyses based on singular-value analysis of MIMO systems may have similar limitations. Nominal settling time did not give a good indication of the likelihood of exceeding settling-time limit, principally because most nominal values already exceeded the limit. Although this result may be an artifact of the settling-time definition (T_S^*), it reveals the counterintuitive result that uncertainty may improve the probability of remaining within a predefined limit. The relationship between maximum control response to a disturbance impulse and the probability of exceeding a control limit is more direct, as most nominal values were about half the limit value. Stochastic root loci and parameter histograms provide insight about the likely positions of the closed-loop roots and the parameter variations that lead to instability, and they suggest ways of improving plant and controller design.

Appendix: Transfer Functions of the Ten Compensators

Design A:

$$\frac{40.42(s+2.388)(s+0.350)}{(s+163.77)\left[s^2+2(0.501)(0.924)s+(0.924)^2\right]}$$

Design B:

$$\frac{42.78(s-1.306)(s+0.1988)}{(s+73.073)\left[s^2+2(0.502)(1.182)s+(1.182)^2\right]}$$

Design C:

$$\frac{0.599(s-1.253)(s+0.1988)}{\left[s^2+2(0.502)(1.182)s+(1.182)^2\right]}$$

Design D:

$$\frac{19881(s+100)(s+0.212)\left[s^2+2(0.173)(0.733)s+(0.733)^2\right]}{\left[s^2+2(0.997)(51.16)s+(51.16)^2\right]\left[s^2+2(0.838)(16.44)s+(16.44)^2\right]}$$

Design E:

$$\frac{5.369(s-0.348)(s+0.0929)}{\left[s^2+2(0.832)(2.21)s+(2.21)^2\right]}$$

Design F:

$$\frac{2246.3(s+0.237)\left[s^2-2(0.32)(1.064)s+(1.064)^2\right]}{(s+33.19)(s+11.79)\left[s^2+2(0.90)(2.75)s+(2.75)^2\right]}$$

Design G:

$$\frac{4430(s+0.08)(s-0.44)(s-2.83)\left[s^2-2(0.102)(0.49)s+(0.49)^2\right]}{\left\{\left[s^2+2(0.70)(11.17)s+(11.17)^2\right]\left[s^2+2(0.89)(3.67)s+(3.67)^2\right]\left[s^2+2(0.29)(3.11)s+(3.11)^2\right]\left[s^2+(0.5)^2\right]\right\}}$$

Design H:

$$\frac{2.13(s+0.145)(s-0.98)(s+3.43)}{\left[s^2+2(0.82)(1.59)s+(1.59)^2\right]\left[s^2+2(0.46)(2.24)s+(2.24)^2\right]}$$

Design I:

$$\frac{16.1(s+0.134)(s-1.174)(s+1.46)}{\left[s^2+2(0.82)(1.05)s+(1.05)^2\right]\left[s^2+2(0.5)(2.18)s+(2.18)^2\right]}$$

Design J:

$$\frac{51.47(s+0.06)(s-0.21)(s+5.41)\left[s^2-2(0.07)(0.51)s+(0.51)^2\right]}{\left[s^2+2(0.72)(2.05)s+(2.05)^2\right]\left[s^2+2(0.68)(5.21)s+(5.21)^2\right]\left[s^2+(0.5)^2\right]}$$

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